

Faddeev approach to the octet and decuplet baryons

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Abstract

A relativistic Faddeev model for the baryon octet is extended to treat the baryon decuplet. We find that after determining the model parameters in the mesonic sector the masses of both nucleon and delta deviate by less than 5% from the experimental data and show only a very weak dependence on the constituent quark mass.

Recently relativistic Faddeev quark-models have been developed for the nucleon [1-6]. So far, the Faddeev approach was applied to the baryon octet only. In this communication, we extend the Faddeev approach to the baryon decuplet as well. The aim of our investigation is to check whether a consistent description of mesonic and baryonic properties is possible within this framework.

The relativistic Faddeev models are especially suited to investigate diquarks [7-9] inside the baryons, since diquark substructures appear naturally in these approaches. The amount of correlation is determined by the baryon properties. Therefore the appearance of bound diquark states is not essential.

Quark models of the nucleon have to respect chiral symmetry. The Nambu–Jona-Lasinio (NJL) model [10] provides a particularly simple implementation of chiral symmetry as it is reviewed in refs. [11, 12]. As most relativistic Faddeev models we therefore start from a NJL-type Lagrangian:

$$\mathcal{L} = \bar{q}(i\partial - m_0)q - G_1(\bar{q}\tilde{\Lambda}_1^{\mu k}q)(\bar{q}\tilde{\Lambda}_{\mu 1}^kq) + G_2(\bar{q}\tilde{\Lambda}_2^{\mu k}q)(\bar{q}\tilde{\Lambda}_{\mu 2}^kq) , \quad (1)$$

where $m_0 = \text{diag}(m_0^u, m_0^d, m_0^s)$, $\tilde{\Lambda}_1^\mu = \gamma^\mu \otimes I_f \otimes \lambda_c^k$ and $\tilde{\Lambda}_2^\mu = \gamma^\mu \gamma_5 \otimes I_f \otimes \lambda_c^k$, λ_c^k being the Gell-Mann matrices.

Using the Fierz transformation the interaction can be expressed in terms of a particle–antiparticle interaction with suitable colour channels

$$\mathcal{L}_I^M = \frac{g_1^a}{2}(\bar{q}\Lambda^{[1]}q)(\bar{q}\Lambda^{[1]}q) + (\text{colour} - \text{octet} - \text{terms}) , \quad (2)$$

where $\Lambda^{[1]} = I_c^\gamma \otimes \lambda_f^k \otimes \Gamma^a$ and a particle–particle interaction

$$\mathcal{L}_I^D = -\frac{g_2^a}{2}(\bar{q}\Lambda^{[3]}C\bar{q}^T)(q^TC^{-1}\Lambda^{[3]}q) + (\text{colour} - \text{sextet} - \text{terms}) , \quad (3)$$

where $\Lambda^{[3]} = t_c^\gamma \otimes \lambda_f^k \otimes \Gamma^a$, with $\Gamma_s^a \in \{I, i\gamma_5\}$ and $\Gamma_v^a \in \{i\gamma^\mu, i\gamma^\mu\gamma_5\}$. The color antitriplet is generated by $t_{\alpha\beta}^\gamma = i\sqrt{\frac{3}{2}}\epsilon_{\alpha\beta\gamma}$. The generators of the flavor

group $U(3)_F$ are denoted λ_F^A by $A \in \{0, \dots, 8\}$. The coupling constants g_1 and g_2 are functions of G_1 and G_2 : $g_1^s = \frac{16}{9}(G_1 + G_2)$, $g_i^v = \frac{1}{2}g_i^s$ and $g_1^a = \frac{1}{2}g_2^a$. For $G_2 = 0$ the above Lagrangian can be viewed as the Fierz transformation of the point like interaction of two color octet currents [1]. The investigation of meson properties in the NJL model, however, proved that the scalar and vector coupling constants have to be varied independently [13]. The recent work of Ishii et al. [5] has shown that the scaling factor $\alpha = \frac{g_1^a}{g_2^a} = \frac{1}{2}$ mentioned above produces a proton mass quite close to the experimental one provided the axial vector diquark is considered in addition to the scalar diquark.

We take into account only the two low lying diquarks namely the scalar one ($\Gamma_s = i\gamma_5$) and the axial vector one ($\Gamma_v^\mu = i\gamma^\mu$).

The two-body T-matrices are calculated in the ladder approximation. For the quark-quark sector this yields:

$$T(q^2) = \tilde{T}(q^2)[\lambda_F^A \otimes t^\gamma \otimes C^{-1}\Gamma^\mu]\mathcal{O}_{\mu\nu}[\lambda_F^A \otimes t^\gamma \otimes \Gamma^\nu C] \quad (4)$$

with $\tilde{T}(q^2) = \left(\frac{1}{2g_2} - \frac{1}{2}J_D(q^2)\right)^{-1}$, $\mathcal{O}_{\mu\nu}$ as the vector structure of the two-body Greens function ($\mathcal{O} = I$ for the scalar diquark, $\mathcal{O}_{\mu\nu} = g_{\mu\nu} - q_\mu q_\nu / q^2$ for the axial vector diquark) and

$$J_D(q^2) = -i \int \frac{d_4 k}{(2\pi)^4} \text{tr}(\Lambda^{[3]} S_F(-q_1) \Lambda^{[3]} S_F(q_2)) . \quad (5)$$

where $q = q_1 + q_2$ and we used $C^{-1}S_F^T(q)C = S_F(-q)$. Since the quarks are treated as identical particles the Pauli principle requires the diquark vertices to be antisymmetric [11]. For the mesons we get $\tilde{T}_M(q^2) = \left(\frac{1}{g_1} - J_M(q^2)\right)^{-1}$ and

$$J_M(q^2) = i \int \frac{d_4 k}{(2\pi)^4} \text{tr}(\Lambda^{[1]} S_F(-q_1) \Lambda^{[1]} S_F(q_2)) . \quad (6)$$

The Lagrangian leads to both bound diquarks and bound mesons, although the corresponding terms appear with different sign in eqn. (2). This difference is compensated by an additional minus sign in the meson polarization function (cf. eqn. (6) and eqn. (5)) created by its closed fermion loop. The two-body T-matrices are diagonal not only in color and flavor space, but also do not mix channels due to the traces appearing in eqns. (5) and (6).

A detailed derivation of the relativistic Faddeev equations is given in refs. [1, 6]. One has to use the same kind of approximation as has already been used in the meson case. The essential one is to neglect the three-body irreducible graphs. This reduces the three-body problem to an effective two-body Bethe-Salpeter equation with an energy dependent interaction.

The Faddeev amplitude therefore reads

$$\mathcal{T}(p_1, p_2, p_3)_{\alpha\beta\gamma} = (C^{-1}t^\gamma \lambda^A \gamma_5)_{\alpha\beta} \tilde{T}(p_1 + p_2)^{AA} \Psi(P, p_3)_\gamma^A$$

$$\begin{aligned}
& + (C^{-1} t^\gamma \lambda^S \gamma^\mu)_{\alpha\beta} \tilde{T}(p_1 + p_2)_{\mu\nu}^{SS} \Psi(P, p_3)_\gamma^{S\ \nu} \\
& + (cyclic) ,
\end{aligned} \tag{7}$$

with $\lambda^{A,S}$ being the antisymmetric and symmetric Gell-Mann matrices respectively. Here *cyclic* means cyclic permutation of all types of indices at once. The amplitudes Ψ are the solutions of the following integral equation:

$$\Psi(P, q)_{(\mu)}^A = i \int \frac{d_4 k}{(2\pi)^4} L(k_1, k_2, P)_{(\mu\tau)}^{AC} \Psi(P, P - k)^{C\ (\tau)} , \tag{8}$$

where $k_1 = q - k$, $k_2 = k$ and

$$L(q, p, l)_{(\mu\tau)}^{AC} = C^{-1} \Lambda^{B\ (\nu)} S_F(q) \Lambda_{(\mu)}^A C S_F(p)^T \tilde{T}(l - p)_{(\nu\tau)}^{BC} . \tag{9}$$

The brackets around the Lorentz indices indicate that they only appear, if the axial vector diquark is involved.

As was observed in ref. [2] the color part of L can be written as a sum of two projectors ($t^B t^A = -3P^{[1]} + \frac{3}{2}P^{[8]}$). This means that no mixing between the color singlet and the color octet occurs. Due to their different signs only one of the possible color multiplets leads to bound states. In our case this is the color singlet. Therefore in the following we only have to consider color singlet states. As all the indices are fixed here, the color indices are omitted in the following.

The next step is to project out from eqn. (7) the flavor multiplets assuming exact $SU(3)_F$. The suitable projectors can in general be constructed by the tools provided in ref. [14]. The projector $P^{[10]}$ on the decuplet symmetrizes all indices, the singlet projector $P^{[1]}$ antisymmetrizes and $P^{[8]} = id - P^{[10]} - P^{[1]}$. The subspace of the octet obtained by this construction contains both types of octets appearing in the decomposition of the flavor content of the three quarks. At this stage it is crucial to observe that the amplitudes Ψ only appear in certain combinations in the equations derived by the above procedure when the explicit representation of the Gell-Mann matrices is introduced. These linear combinations are respected by the flavor content of L . Therefore this construction provides the basis that block diagonalizes L (table (1)). Since we know the flavor structure of every block it is possible to uniquely identify the particle described by a special subspace.

Within this basis we obtain the following set of equations:

for the singlet:

$$^{[1]}\Phi^s = -2A^{(ss)}\ ^{[1]}\Phi^s , \tag{10}$$

for the octet:

$$\begin{aligned}
^{[8]}\Phi^{a\ \mu} &= -A^{(aa)\mu}_{\ \nu} ^{[8]}\Phi^{a\ \nu} - \sqrt{3}A^{(as)\mu}_{\ \nu} ^{[8]}\Phi^s \\
^{[8]}\Phi^s &= -\sqrt{3}A^{(sa)}_{\ \nu} ^{[8]}\Phi^{a\ \nu} + A^{(ss)} ^{[8]}\Phi^s ,
\end{aligned} \tag{11}$$

and for the decuplet:

$$^{[10]}\Phi^\mu = 2A^{(aa)\mu}_{\ \nu} ^{[10]}\Phi^\nu , \tag{12}$$

Table 1: Quark content and basis for baryon amplitudes.

	$[8] \Phi^{a\nu}$	$[8] \Phi^s$
p (uud)	$\frac{1}{\sqrt{3}} (\Psi_u^{a(u d)\nu} - \sqrt{2} \Psi_d^{a(u u)\nu})$	$i \Psi_u^{s(u d)}$
n (udd)	$\frac{1}{\sqrt{3}} (\Psi_d^{a(u d)\nu} - \sqrt{2} \Psi_u^{a(d d)\nu})$	$i \Psi_d^{s(u d)}$
Σ^+ (uus)	$\frac{1}{\sqrt{3}} (\Psi_u^{a(u s)\nu} - \sqrt{2} \Psi_s^{a(u u)\nu})$	$i \Psi_u^{s(u s)}$
Σ^o (uds)	$\frac{1}{\sqrt{6}} (\Psi_u^{a(d s)\nu} + \Psi_d^{a(u s)\nu} - 2 \Psi_s^{a(u d)\nu})$	$\frac{i}{\sqrt{2}} (\Psi_d^{s(u s)} + \Psi_u^{s(d s)})$
Σ^- (dds)	$\frac{1}{\sqrt{3}} (\Psi_d^{a(d s)\nu} - \sqrt{2} \Psi_s^{a(d d)\nu})$	$i \Psi_d^{s(d s)}$
Ξ^o (uss)	$\frac{1}{\sqrt{3}} (\Psi_s^{a(u s)\nu} - \sqrt{2} \Psi_u^{a(s s)\nu})$	$i \Psi_s^{s(u s)}$
Ξ^- (dss)	$\frac{1}{\sqrt{3}} (\Psi_s^{a(d s)\nu} - \sqrt{2} \Psi_d^{a(s s)\nu})$	$i \Psi_s^{s(d s)}$
Λ (uds)	$\frac{1}{\sqrt{2}} (\Psi_u^{a(d s)\nu} - \Psi_d^{a(u s)\nu})$	$\frac{i}{\sqrt{6}} (2 \Psi_s^{s(u d)} - \Psi_u^{s(d s)} + \Psi_d^{s(u s)})$
	$[10] \Phi^{a\nu}$	
Δ^{++} (uuu)	$\Psi_u^{a(u u)\nu}$	
Δ^+ (uud)	$\frac{1}{\sqrt{3}} (\Psi_d^{a(u u)\nu} + \sqrt{2} \Psi_u^{a(u d)\nu})$	
Δ^o (udd)	$\frac{1}{\sqrt{3}} (\Psi_u^{a(d d)\nu} + \sqrt{2} \Psi_d^{a(u d)\nu})$	
Δ^- (ddd)	$\Psi_d^{a(d d)\nu}$	
Σ^{+*} (uus)	$\frac{1}{\sqrt{3}} (\Psi_s^{a(u u)\nu} + \sqrt{2} \Psi_u^{a(u s)\nu})$	
Σ^{o*} (uds)	$\frac{1}{\sqrt{3}} (\Psi_s^{a(u d)\nu} + \Psi_u^{a(d s)\nu} + \Psi_d^{a(s u)\nu})$	
Σ^{-*} (dds)	$\frac{1}{\sqrt{3}} (\Psi_s^{a(d d)\nu} + \sqrt{2} \Psi_d^{a(d s)\nu})$	
Ξ^{o*} (uss)	$\frac{1}{\sqrt{3}} (\Psi_u^{a(s s)\nu} + \sqrt{2} \Psi_s^{a(s u)\nu})$	
Ξ^{-*} (dss)	$\frac{1}{\sqrt{3}} (\Psi_d^{a(s s)\nu} + \sqrt{2} \Psi_s^{a(d s)\nu})$	
Ω^- (sss)	$\Psi_s^{a(s s)\nu}$	
		$[1] \Phi^a$
- (uds)		$\frac{i}{\sqrt{3}} (-\Psi_s^{s(u d)} - \Psi_u^{s(d s)} + \Psi_d^{s(s u)})$

where A is a linear operator defined in the following way

$$(A^{AB}\Phi^B)(P, q) := i \int \frac{d_4 k}{(2\pi)^4} L_D(q - k, k, P)_{(\mu\tau)}^{AB} \Phi(P, P - k)^B{}^{(\tau)}. \quad (13)$$

Here L_D is the Dirac part of the matrix L defined in (9) and a and s stand for a suitable axial vector or scalar diquark respectively.

Eqns. (10)–(12) now in principle can be solved numerically. Given the relatively good performance of the static approximation [4, 5], we simplify the numerical work by replacing the propagator of the exchanged quark by $S_F(q) \rightarrow -\frac{1}{M}$ following Buck et al. [2]. The scattering kernel L is therefore independent of the outgoing diquark momentum q which totally disappears from eqn. (8). Consequently the amplitudes Φ only depend on the baryon momentum P . A is no longer an integral operator but a simple matrix. The remaining integral has to be regularized. We used a sharp Euclidean cut off Λ . The treatment of the singularities in the integral required to perform a Wick rotation is discussed in [4]. To ensure good spin of the solutions we used the spin projectors given in reference [15].

The equations (10)–(12) are valid only in the limit of $SU(3)_F$ symmetry. If one introduces different quark masses, the operator A defined in equation (13) receives additional contributions which couple the equations for octet and decuplet. As long as we keep exact $SU(2)_F$ the equation for the Λ is mixed with the flavour singlet only and there is no mixing for the nucleon and the delta. Using the spin projectors for spin- $\frac{3}{2}$ and spin- $\frac{1}{2}$ we get two sets of coupled equations. The solutions can be naturally identified with the observables.

The model contains 5 parameters: 2 current masses ($m_0^u = m_0^d$, m_0^s), the cut off (Λ) and 2 coupling constants (g_1^s , g_1^v). On the other hand we have 14 observables:

- 4 masses from the baryon decuplet (M_Δ , M_{Σ^*} , M_{Ξ^*} , M_{Ω^-}),
- 4 masses from the baryon octet (M_N , M_Σ , M_Λ , M_Ξ),
- 3 masses from the nonet of the vector mesons ($m_\rho = m_\omega$, m_{K^*} , m_Φ), where the Φ is treated as pure $s\bar{s}$ state,
- 2 masses from the nonet of the pseudoscalar mesons (m_π , m_K) (To calculate the mass of the η' -meson one would have to include the t'Hooft interaction [11]) and
- the pion decay constant f_π using the Gell-Mann–Oaks–Renner relation [16].

For a given Cut-off we can therefore fix all 4 parameters using only mesonic data. We investigate the behavior of the spectrum in dependence of the constituent mass of the up quark. The cut off is determined by the NJL gap eqn. [10]. Note that we have the constraint of $m_u^* \geq 411$ MeV in order to bind the $\Delta_{33}(1232)$.

As a check of our formalism, we switched off the vector coupling in the particle-particle channel ($g_2^v = 0$) and reproduced the results of Buck et al. [2],

provided the scalar coupling is increased to reproduce the nucleon mass. In this case a strongly bound scalar diquark is found ($m_{s(u\bar{d})} = 605$ MeV with a binding energy of $E_{s(u\bar{d})} = 295$ MeV for $m_u^* = 450$ MeV). We confirm the finding of [5], that if both scalar and axial vector diquark channels are considered simultaneously, the binding energy of the three-body system is increased. Therefore a less bound scalar diquark is required for the binding of the nucleon. A similar result was found in a recent analysis of lattice gauge calculations [17].

For constituent quark masses below 440 MeV we had no bound axial vector diquarks, but for larger masses binding occurred.

The deviation of the masses of decuplet and octet baryons calculated in our model from the experimental values is shown in figure 1 as a function of the quark mass. The masses of both nucleon and $\Delta_{33}(1232)$ are reproduced within 5 % for quark masses between 411 MeV and 800 MeV. With increasing quark masses the coupling constants are adjusted so that the meson masses remain constant. This suffices to produce both a nucleon and a delta mass which show only weak dependence on the constituent quark mass.

In the solitonic approach to the baryons of Weigel et al. [18], the best mass splittings were obtained with a constituent quark mass of 390 MeV. In our model the fit quality is best for the smallest quark mass of 411 MeV, but unfortunately we cannot further decrease the quark mass, since our model lacks a confinement mechanism.

For a constituent quark mass of 411 MeV, the masses of the strange baryons deviate from the experimental ones by approximately 15 % (table (2)) and one can not improve the agreement with the data by increasing the quark masses. We checked that treating the scaling factor $\alpha = \frac{g_1^a}{g_2^a}$ as a free parameter reduces the maximal deviation from the experimental masses to 7 %.

The fact that the strange baryons deviate more strongly from the data than the proton and the delta may be taken as a hint that the t'Hooft interaction should not be neglected. The t'Hooft interaction affects the scalar diquarks only and therefore only the octet baryons.

To summarize, we have shown that the Faddeev approach to the baryonic sector of the Nambu–Jona-Lasinio model is able to provide a consistent description of both octet and decuplet baryons.

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Figure 1: The deviation of the numerical results from the experimental data in percent.

Table 2: The numerical results for the baryon masses for a fixed constituent mass $m_u^* = 411$ MeV. Deviations from the experimental values in percent are added in brackets. Data used to fix the parameters are starred (*). For the constituent mass choosen the axial vector diquarks remained unbound and are therefore not mentioned.

	M [MeV]	E^b [MeV]
quarks		
m_0^u	8.4	—
m_0^s	204	—
m_s^*	617	—
mesons		
f_π	93*	—
π	135*	687
K	495*	533
ρ	770*	52
K^*	938 (5)	105
Φ	1096 (7)	138
diquarks		
$s_{(ud)}$	710	112
$s_{(us)}$	899	129
baryons		
N	901 (4)	220
Σ	1081 (9)	229
Λ	941 (16)	369
Ξ	1114 (15)	402
Δ	1212 (2)	21
Σ^*	1401 (1)	38
Ξ^*	1597 (4)	48
Ω^-	1801 (8)	50

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